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On cyclic edge-connectivity of transitive graphs

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ABSTRACT

A cyclic edge-cut of a graph G is an edge set, the removal of which separates two cycles. If G has a cyclic edge-cut, then it is said to be cyclically separable. For a cyclically separable graph G, the cyclic edge-connectivity $c\lambda(G)$ is the cardinality of a minimum cyclic edge-cut of G. In this paper, we first prove that for any cyclically separable graph G, $c\lambda(G) \le \zeta(G) = \min\{\omega(X) \mid X \text{ induces a shortest cycle in } G\}$, where $\omega(X)$ is the number of edges with one end in X and the other end in $V(G) \setminus X$. A cyclically separable graph G with $c\lambda(G) = \zeta(G)$ is said to be cyclically optimal. The main results in this paper are: any connected k-regular vertex-transitive graph with $k \ge 4$ and girth at least 5 is cyclically optimal; any connected edge-transitive graph with minimum degree at least 4 and order at least 6 is cyclically optimal.

1. Introduction

Let $G = (V, E)$ be a simple graph and F be a set of edges in G. Call F a *cyclic edge-cut* if $G - F$ is disconnected and at least two of its components contain cycles. Clearly, a graph has a cyclic edge-cut if and only if it has two disjoint cycles. Lovász [8] characterized all multigraphs without two disjoint cycles. The characterization can also be found in [2]. We call those graphs which do have cyclic edge-cuts *cyclically separable*. Following [13], we define the *cyclic edge-connectivity of* G, denoted by $c\lambda(G)$, as follows: if G is not connected, then $c\lambda(G) = 0$; if G is connected but does not have two disjoint cycles, then $c\lambda(G) = \infty$; otherwise, $c\lambda(G)$ is the minimum cardinality over all cyclic edge-cuts of G. The definition of cyclic vertex-connectivity can be found in [14].

Cyclic edge-connectivity plays an important role in many classic fields of graph theory. Most of the previous works focus on using the value of $c\lambda$ as a condition to conquer other problems such as in studying integer flow conjectures [19]. In fact, $c\lambda$ can also be used as a measure of network reliability. The classic measure of network reliability is the edge-connectivity $\lambda(G)$ and/or the vertex connectivity $\kappa(G)$. In general, the larger $\lambda(G)$ and/or $\kappa(G)$ are, the more reliable the network is. However, $\lambda(G)$ and $\kappa(G)$ are worst-case measures. For example, in an n-dimensional hypercube Q_n, the edge-connectivity is n. When n edges break down in Q_n, the probability that these failures are exactly the set of edges incident with a common vertex is $2^n / \binom{n \cdot 2^{n-1}}{n}$, which is very small. Hence $\lambda(G)$ and/or $\kappa(G)$ underestimate the resilience of a network [11]. To overcome such a shortcoming, Latifi et al. [6] proposed a kind of conditional edge-connectivity [5], denoted by $\lambda^k(G)$, which is the minimum size of an edge-cut S such that each vertex has degree at least k in $G - S$. When $k = 2$, and the minimum degree $\delta(G) \ge 3$, it can be seen that $\lambda^2(G) = c\lambda(G)$. In fact, since every subgraph of G with minimum degree at least 2 has a cycle, we see that $c\lambda(G) \le \lambda^2(G)$. On the other hand, let S be a minimum cyclic edge-cut of G. By the minimality of S, $G - S$ has exactly two connected components. If a component of $G - S$ has a degree-one vertex, then moving it to the other component decreases the number of edges in the minimum cyclic edge-cut, a contradiction. Hence S is also a λ^2-cut, and thus $\lambda^2(G) \le c\lambda(G)$.

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As in the case of $\lambda(G)$ and $\kappa(G)$, we expect $c\lambda(G)$ to be as large as possible. For two vertex sets $X, Y \subseteq V$, $[X, Y]_G$ is the set of edges with one end in X and the other end in Y. $G[X]$ is the subgraph of G induced by vertex set X, $\overline{X}$ is the complement of X, $\omega_G(X) = |[X, \overline{X}]_G|$ is the number of edges between X and $\overline{X}$ in G. When the graph under consideration is obvious, we omit the subscription G and use the notation $[X, Y]$, $\omega(X)$ etc. If $[X, \overline{X}]$ is a minimum cyclic edge-cut, then both $G[X]$ and $G[\overline{X}]$ are connected. Define

$$\zeta(G) = \min\{\omega(X) \mid X \text{ induces a shortest cycle in } G\}.$$

We shall show that any cyclically separable graph G satisfies $c\lambda(G) \le \zeta(G)$. Hence, a cyclically separable graph G is called *cyclically optimal* if $c\lambda(G) = \zeta(G)$. In this paper, we study the cyclically optimality of vertex-transitive graphs and edge-transitive graphs.

Some previous studies in this line include [7,12,18]. In [7], Lou and Wang gave a polynomial algorithm deciding whether $c\lambda(G) < \infty$ for multigraph G. In [12], Nedela and Skoviera studied the existence of cyclic edge-cuts in cubic multigraphs, showing that a connected cubic graph G has no cyclic edge-cut if and only if it is isomorphic to one of K_4, $K_{3,3}$ or Θ_2 (the multigraph with two vertices and three edges between them), furthermore, $c\lambda(G) \le \zeta(G)$ in this case. Xu and Liu [18] showed that a k-regular simple graph G with $k \ge 3$ which is not K_4, K_5, and $K_{3,3}$ is cyclically separable, and $c\lambda(G) \le \zeta(G)$. Furthermore, they proved that a connected k-regular vertex-transitive graph G with $k \ge 4$, $k \ne 5$ and girth $g(G) \ge 5$ is cyclically optimal. It should be pointed out that Xu and Liu's work is in fact on $\lambda^2(G)$. In view of the discussion in the second paragraph of this paper, we just state their results using our present terminology.

In this paper, we only consider simple connected graphs with order at least 6 (since no simple graph with fewer than six vertices has a cyclic edge-cut). We first show that $c\lambda(G) \le \zeta(G)$ holds for any cyclically separable graph G. Then we prove that any vertex-transitive graph with regularity degree $k \ge 4$ and girth $g(G) \ge 5$ is cyclically optimal. This is an improvement on Xu and Liu's result [18] since by our proof, vertex-transitive graphs with regularity 5 are also cyclically optimal. Finally, we prove that any edge-transitive graph G with minimum degree $\delta(G) \ge 4$ is cyclically optimal.

We follow [3] for terminology and notation not defined here.

2. Upper bound on $c\lambda(G)$

In this section, we prove $c\lambda(G) \le \zeta(G)$ for any cyclically separable graph G.

Sometimes we use the symbol for a graph to represent its vertex set. For example, if G_1 is a subgraph of G, $\omega(G_1)$ is used instead of $\omega(V(G_1))$. Sometimes, we also use $\overline{G}_1$ to represent the subgraph of G induced by $\overline{V(G_1)}$.

Theorem 2.1. *Any cyclically separable graph G satisfies* $c\lambda(G) \le \zeta(G)$.

Proof. Let $[X, \overline{X}]$ be a minimum cyclic edge-cut. Since both $G[X]$ and $G[\overline{X}]$ are connected and have cycles, we have $|E(G[X])| \ge |X|$ and $|E(G[\overline{X}])| \ge |\overline{X}|$. Then

$$|E(G)| = |E(G[X])| + |E(G[\overline{X}])| + \omega(X) \ge |X| + |\overline{X}| + c\lambda(G) = |V(G)| + c\lambda(G),$$

whence $c\lambda(G) \le \omega(X)$. Let C be a shortest cycle in G such that $\omega(C) = \zeta(G)$. If $G[\overline{C}]$ contains a cycle, then $c\lambda(G) \le \omega(C) = \zeta(G)$. Hence we assume that $G[\overline{C}]$ is a forest. Then $|E(\overline{C})| \le |V(\overline{C})| - 1$, and thus

$$\begin{aligned} \zeta(G) = \omega(C) &= |E(G)| - |E(C)| - |E(\overline{C})| \\ &\ge |E(G)| - |V(C)| - (|V(\overline{C})| - 1) \\ &= |E(G)| - |V(G)| + 1 > c\lambda(G). \end{aligned} \tag{1}$$

The theorem is proved. □

3. Some preliminaries

In this section, we introduce some terminologies and some basic results which will be used in the following sections.

In this paper we only consider simple graphs with minimum degree at least 3. In this case, the following characterization of the existence of two disjoint cycles can be found in [9].

Theorem 3.1 (*[9]*). *Let G be a simple graph with all degrees at least 3 which contains no two disjoint cycles. Then G is one of the following graphs:* (i) K_5, (ii) *a wheel,* (iii) $K_{3,n-3}$ *with any set of edges connecting vertices in the 3-element class added.*

As a corollary of Theorem 3.1, we have the following result.

Corollary 3.2. *Let G be a simple connected graph with* $\delta(G) \ge 3$ *and* $g(G) \ge 5$ *or* $\delta(G) \ge 4$ *and order* $n \ge 6$. *Then G is cyclically separable.*

A vertex set X is a *cyclic edge-fragment*, if $[X, \overline{X}]$ is a minimum cyclic edge-cut. A cyclic edge-fragment with the minimum cardinality is called a *cyclic edge-atom*. Whenever no confusion arises, *fragment* will stand for cyclic edge-fragment and *atom* will stand for cyclic edge-atom. Clearly, if X is a fragment, then $\overline{X}$ is also a fragment, and both $G[X]$ and $G[\overline{X}]$ are connected. The following observation will be used frequently in the proofs: If X is an atom, and X' is a proper subset of X such that $[X', \overline{X'}]$ is a cyclic edge-cut, then

$$\omega(X') > \omega(X). \tag{2}$$

The concepts of fragment and atom were first proposed by Mader [10] and Watkins [16], and their variations play an important role in studying various kinds of connectivity. An atom is said to be *trivial*, if it induces a cycle of G, otherwise it is *non-trivial*.

For a vertex u and a vertex set X, $N_X(u) = \{v \in X \mid v \text{ is adjacent with } u\}$ is the set of neighbors of u in X. Denote by $d_X(u) = |N_X(u)|$. In particular, if $u \in X$ and $X = V(G_1)$ for some subgraph G_1 of G, then $d_X(u) = d_{G_1}(u)$ is exactly the degree of u in G_1. It should be noted that in this paper, we may come across other cases in which $d_X(u)$ is merely the number of neighbors of u in X instead of the degree of u in a subgraph.

Lemma 3.3. *Let G be a connected graph with $\delta(G) \geq 3$ and X be a fragment. Then*

(i) $\delta(G[X]) \geq 2$*;*
(ii) *If $\delta(G[X]) \geq 3$, then $d_X(v) \geq d_{\overline{X}}(v)$ holds for any $v \in X$;*
(iii) *If $G[X]$ is not a cycle and v is a vertex in X with $d_X(v) = 2$, then $d_X(v) \geq d_{\overline{X}}(v)$.*
(iv) *If $\delta(G) \geq 4$, and X is a non-trivial atom of G, then $\delta(G[X]) \geq 3$. Furthermore, $d_X(v) > d_{\overline{X}}(v)$ holds for any $v \in X$.*

Proof. (i) Suppose there is a vertex $v \in X$ with $d_X(v) = 1$. Then $G[X - v]$ contains a cycle, and thus $[X - v, \overline{X} + v]$ is a cyclic edge-cut. But

$$\omega(X - v) = \omega(X) - d_{\overline{X}}(v) + d_X(v) \leq \omega(X) - \delta(G) + 2 < \omega(X) = c\lambda(G),$$

a contradiction to the definition of $c\lambda(G)$. Hence $\delta(G[X]) \geq 2$.

(ii) Assume there exists a vertex $v \in X$ with $d_X(v) < d_{\overline{X}}(v)$. Since $\delta(G[X]) \geq 3$, we have $\delta(G[X - v]) \geq \delta(G[X]) - 1 \geq 2$. Hence $G[X - v]$ contains a cycle, and thus $[X - v, \overline{X} + v]$ is a cyclic edge-cut. But

$$\omega(X - v) = \omega(X) - d_{\overline{X}}(v) + d_X(v) < \omega(X) = c\lambda(G),$$

a contradiction.

(iii) If $G[X - v]$ is acyclic, then by $\delta(G[X]) \geq 2$ and $d_X(v) = 2$, we see that $G[X - v]$ is a path whose ends are the two neighbors of v in X. It follows that $G[X]$ is a cycle, contradicting the assumption. Hence $G[X - v]$ contains a cycle, and thus $[X - v, \overline{X} + v]$ is a cyclic edge-cut. Suppose $d_X(v) < d_{\overline{X}}(v)$, then

$$c\lambda(G) \leq \omega(X - v) = \omega(X) - d_{\overline{X}}(v) + d_X(v) < \omega(X) = c\lambda(G),$$

a contradiction.

(iv) Suppose there is a vertex $v \in X$ with $d_X(v) = 2$. As in the proof of (iii), using the assumption that X is non-trivial, we see that $[X - v, \overline{X} + v]$ is a cyclic edge-cut. By Observation (2), we have $\omega(X - v) > \omega(X)$. On the other hand,

$$\omega(X - v) = \omega(X) - d_{\overline{X}}(v) + d_X(v) \leq \omega(X) - \delta(G) + 4 \leq \omega(X),$$

a contradiction. Hence $\delta(G[X]) \geq 3$.

Suppose there is a vertex $v \in X$ with $d_X(v) \leq d_{\overline{X}}(v)$. As in the proof of (ii), we see that $[X - v, \overline{X} + v]$ is a cyclic edge-cut with

$$\omega(X - v) = \omega(X) - d_{\overline{X}}(v) + d_X(v) \leq \omega(X).$$

But this contradicts Observation (2). $\square$

A *biregular graph* is a bipartite graph in which all the vertices from the same partite set have the same degree. If the two distinct degrees are p and q respectively ($p \geq q$), then we abbreviate the bipartite graph as a (p, q)-*biregular graph* and use $V_p(G)$ (resp. $V_q(G)$) to denote the set of vertices with degree p (resp. q) in $V(G)$. For a vertex set $X \subseteq V(G)$, denote $V_p(X) = V_p(G) \cap X$ and $V_q(X) = V_q(G) \cap X$.

In the following, we give two results for regular graphs and/or biregular graphs. It is easy to see that for a k-regular graph G, $\zeta(G) = (k - 2)g$, and for a (p, q)-biregular graph G, $\zeta(G) = (p + q - 4)g/2$, where $g = g(G)$ is the girth of the graph.

Lemma 3.4. *Let G be a simple connected graph with girth g, X be a vertex set. Suppose $G[X]$ is a forest. If either*

(i) *G is regular and $|X| \geqslant g$, or*
(ii) *G is biregular and $|V_p(X)| \geqslant g/2$, $|V_q(X)| \geqslant g/2$,*

then $\omega(X) \geq \zeta(G) + 2$.

Proof. Since $G[X]$ is a forest, we have $|E(G[X])| \leqslant |X| - 1$.

(i) Suppose G is k-regular. Then

$$\omega(X) = k|X| - 2|E(G[X])| \geqslant (k-2)|X| + 2 \geq (k-2)g + 2 = \zeta(G) + 2.$$

(ii) Suppose G is a (p, q)-biregular graph. Then

$$\begin{aligned}\omega(X) &= p|V_p(X)| + q|V_q(X)| - 2|E(X)| \\ &\geqslant (p-2)|V_p(X)| + (q-2)|V_q(X)| + 2 \\ &\geq (p+q-4)g/2 + 2 \\ &= \zeta(G) + 2. \quad \square\end{aligned}$$

Lemma 3.5. *Let $G = (V, E)$ be a cyclically separable k-regular graph or (p, q)-biregular graph, and let C be a shortest cycle. Then $[C, \overline{C}]$ is a cyclic edge-cut.*

Proof. If $[C, \overline{C}]$ is not a cyclic edge-cut, then $G[\overline{C}]$ is a forest. We are going to derive a contradiction by taking $X = V(\overline{C})$ in Lemma 3.4. Note that $|V(G)| \geq 2g$ since G contains two disjoint cycles. Hence $|V(\overline{C})| = |V(G)| - |V(C)| \geq g$. In the case that G is (p, q)-biregular, consider two disjoint cycles C_1 and C_2. For $i \in \{1, 2\}$, C_i has exactly half of vertices in $V_p(G)$ (since G is bipartite). Hence $|V_p(C_i)| \geq g/2$ $(i = 1, 2)$, and thus $|V_p(G)| \geq |V_p(C_1)| + |V_p(C_2)| \geq g$. It follows that $|V_p(\overline{C})| = |V_p(G)| - |V_p(C)| \geq g/2$. Since $p \geq q$ (in the definition of a (p, q)-biregular graph), we have $|V_q(G)| \geq |V_p(G)|$, and thus $|V_q(\overline{C})| = |V_q(G)| - |V_q(C)| \geq g/2$. By Lemma 3.4,

$$\omega(\overline{C}) \geq \zeta(G) + 2 > \zeta(G) = \omega(C) = \omega(\overline{C}),$$

which is a contradiction. $\square$

4. Cyclically optimal vertex-transitive graphs

In [18], Xu and Liu proved that a connected vertex-transitive graph G of order $n \geq 7$, girth $g \geq 5$ and degree k ($k \geq 4$ and $k \neq 5$) is cyclically optimal. In this section, we shall prove that under the same conditions, G is also cyclically optimal when $k = 5$. Instead of dealing with the case $k = 5$ separately, we use a uniform method to obtain the declared result for all $k \geq 4$.

We need the following result proved by Xu.

Theorem 4.1 ([17]). *Let G be a connected k-regular graph with $k \geq 3$, which is different from K_4, K_5 and $K_{3,3}$. Then G is cyclically optimal if and only if every atom of G is trivial.*

It should be noted that for a regular graph or a biregular graph, by the definitions of $c\lambda(G)$ and atom, a trivial atom of G must induce a *shortest* cycle.

It should also be noted that *every atom is trivial* is equivalent to *there exists an atom which is trivial*. In fact, suppose G has a trivial atom. Then every atom has g vertices. Let X be an arbitrary atom. Since $G[X]$ contains a cycle, we see that $G[X]$ is a shortest cycle. Hence X is a trivial atom. Then Theorem 4.1 is equivalent to saying that G is not cyclically optimal if and only if every atom is non-trivial.

The first part of the following lemma can also be found in [18].

Lemma 4.2. *Let G be a k-regular graph with $k \geq 3$ and girth g, and X, Y be two distinct atoms with $X \cap Y \neq \emptyset$. If G is not cyclically optimal, then $|X \cap Y| \leq g - 1$ and $|X| = |Y| \leq 2(g-1)$.*

Proof. Suppose $|X \cap Y| \geq g$. By the minimality of atom, we have $|X| = |Y| \leq |V(G)|/2$. Hence,

$$|\overline{X \cup Y}| = |V(G)| - |X| - |Y| + |X \cap Y| \geq |X \cap Y| \geq g.$$

We first show that $\omega(X \cap Y) > c\lambda(G)$. In fact, if $G[X \cap Y]$ contains a cycle, by noting that $G[\overline{X \cap Y}] \supseteq G[\overline{Y}]$ also contains a cycle, we see that $[X \cap Y, \overline{X \cap Y}]$ is a cyclic edge-cut. Hence $\omega(X \cap Y) > \omega(X) = c\lambda(G)$ by Observation (2). If $G[X \cap Y]$ is acyclic, then it follows from Lemma 3.4 and Theorem 2.1 that $\omega(X \cap Y) \geq \zeta(G) + 2 > c\lambda(G)$. By a similar argument applied to $\overline{X \cup Y}$, we see that $\omega(X \cup Y) = \omega(\overline{X \cup Y}) \geq c\lambda(G)$.

Then, by the well-known submodular inequality (see, for example, [1]),

$$2c\lambda(G) < \omega(X \cap Y) + \omega(X \cup Y) \leq \omega(X) + \omega(Y) = 2c\lambda(G),$$

a contradiction. Hence $|X \cap Y| \leq g - 1$.

Suppose $|X| = |Y| > 2(g-1)$, then $|X \cap \overline{Y}| = |\overline{X} \cap Y| \geq g$, and a contradiction can be obtained by a similar argument as above. Hence $|X| = |Y| \leq 2(g-1)$. $\square$

Corollary 4.3. *Let G be a k-regular connected graph with girth $g \geq 5$ and $k \geq 4$. Suppose G is not cyclically optimal. Then for any two distinct atoms X and Y of G, $X \cap Y = \emptyset$.*

Proof. Suppose $X \cap Y \neq \emptyset$. By Lemma 4.2, $|X| \leq 2(g-1)$. By the discussion after Theorem 4.1, X is non-trivial. Since $k \geq 4$, by Lemma 3.3(iv), we have $\delta(G[X]) \geq 3$. Since $g \geq 5$, by Corollary 3.2, $G[X]$ contains two disjoint cycles. But then $|X| \geq 2g$, a contradiction. □

An *imprimitive block* of G is a proper non-empty subset A of $V(G)$ such that for any automorphism $\phi \in Aut(G)$, either $\phi(A) = A$ or $\phi(A) \cap A = \emptyset$. The following results show why the concept of imprimitive block is useful.

Lemma 4.4 ([4,15]). *Let $G = (V, E)$ be a graph and let Y be the subgraph of G induced by an imprimitive block A of G. If G is vertex-transitive, then so is Y. If G is edge-transitive, then A is an independent subset of G.*

Theorem 4.5. *Let G be a connected k-regular vertex-transitive graph with $k \geq 4$ and $g(G) \geq 5$, then G is cyclically optimal.*

Proof. By Corollary 3.2, G is cyclically separable. Suppose G is not cyclically optimal. Then by Corollary 4.3, every atom is an imprimitive block of G.

Let X be an atom of G. By Lemma 4.4, $G[X]$ is vertex-transitive. Suppose the regularity of $G[X]$ is t. Since G is connected, and by Lemma 3.3(iv), then $3 \leq t \leq k-1$. Let C be a shortest cycle of $G[X]$. Since G is simple and $g(G) \geq 5$, C is chordless and no two vertices on C have a common neighbor outside of $V(C)$. Hence $\omega_X(C) \leq |X - V(C)|$. Then

$$\begin{aligned}\omega_G(C) &= \omega_X(C) + (k-t)|V(C)| \\ &\leq |X - V(C)| + (k-t)|V(C)| \\ &\leq (k-t)(|X| - |V(C)|) + (k-t)|V(C)| \\ &= (k-t)|X| = \omega_G(X).\end{aligned}$$

On the other hand, since $[C, \overline{C}]_G$ is a cyclic edge-cut, and C is a proper subset of X, we have $\omega_G(C) > \omega_G(X)$ by Observation (2), a contradiction. □

Remark. The condition $g(G) \geq 5$ in Theorem 4.5 is necessary since there exist graphs with girth 3 or 4 which are not cyclically optimal. The graph $K_m \times K_2$, the Cartesian product of K_m and K_2, is a vertex-transitive graph with girth 3 and regularity m. For $m \geq 4$, $c\lambda(K_m \times K_2) = m < 3(m-2) = \zeta(K_m \times K_2)$. The graph $K_{n,n} \times K_2$ is a vertex-transitive graph with girth 4 and regularity $n+1$. For $n \geq 3$, $c\lambda(K_{n,n} \times K_2) = 2n < 4(n-1) = \zeta(K_{n,n} \times K_2)$.

5. Cyclically optimal edge-transitive graphs

In this section, we prove that all connected edge-transitive graphs with minimum degree at least 4 and order at least 6 are cyclically optimal.

The following fact is well known.

Lemma 5.1 ([1]). *If a connected graph G is edge-transitive but not vertex-transitive, then G is bipartite. Furthermore, vertices in the same partite set are in the same orbit under Aut(G).*

It follows that if an edge-transitive graph is not regular, then it is (p, q)-biregular for some $p > q$. First, we prove a result for (p, q)-biregular graph which is similar to that of Theorem 4.1.

Theorem 5.2. *Let G be a cyclically separable biregular graph. Then G is cyclically optimal if and only if every atom of G is trivial; G is not cyclically optimal if and only if every atom is non-trivial.*

Proof. Let X be an atom of G. Suppose X is trivial. Then $c\lambda(G) = \omega(X) \geq \zeta(G)$. Since we have proved in Theorem 2.1 that $c\lambda(G) \leq \zeta(G)$, we have $c\lambda(G) = \zeta(G)$. Hence G is cyclically optimal.

Conversely, suppose G is cyclically optimal. Let C be a shortest cycle with $\omega(C) = \zeta(G)$. Then $\omega(C) = c\lambda(G)$. By Lemma 3.5, $[C, \overline{C}]$ is a cyclic edge-cut. Hence $V(C)$ is a fragment. Since every atom of G contains at least $g(G)$ vertices, we see that C is a trivial atom. By an argument similar to that after Theorem 4.1, every atom of G is trivial. □

A fragment X of a (p, q)-biregular graph G is said to be *good* if both $|V_p(X)| \geq g(G)$ and $|V_q(X)| \geq g(G)$. Next, we shall show that two distinct good atoms are mutually disjoint.

Lemma 5.3. *Let G be a cyclic non-optimal (p, q)-biregular graph, and X be a good fragment of G. If X_1, X_2 is a partition of X (that is, if $X = X_1 \cup X_2$ and $X_1 \cap X_2 = \emptyset$), and $G[X_1]$ and $G[X_2]$ are acyclic, then either $\omega(X_1) > c\lambda(G)$ or $\omega(X_2) > c\lambda(G)$.*

Proof. By assumption, $|E(G[X_1])| \leq |X_1| - 1$ and $|E(G[X_2])| \leq |X_2| - 1$. Hence

$$\begin{aligned}\omega(X_1) + \omega(X_2) &= p|V_p(X)| + q|V_q(X)| - 2(E(G[X_1]) + E(G[X_2])) \\ &\geq (p-2)|V_p(X)| + (q-2)|V_q(X)| + 4 \\ &> (p+q-4)g \\ &= 2\zeta(G) > 2c\lambda(G).\end{aligned}$$

The result follows. □

Lemma 5.4. *Let G be a cyclically separable (p, q)-biregular graph with $p > q \geq 4$. Suppose G is not cyclically optimal and X, Y are two distinct good atoms of G. Then $X \cap Y = \emptyset$.*

Proof. By Theorem 5.2, X and Y are non-trivial. Suppose $X \cap Y \neq \emptyset$, we shall derive a contradiction. Denote by $A = X \cap Y$, $B = X \cap \overline{Y}$, $C = \overline{X} \cap Y$, and $D = \overline{X} \cap \overline{Y}$. Since $X \neq Y$, A is a proper subset of X. Assume, without loss of generality, that

$$|[A, B]| \leq |[A, C]|. \tag{3}$$

Then

$$\begin{aligned}\omega(B) &= \omega(X) - |[A, \overline{X}]| + |[A, B]| \\ &\leq \omega(X) - |[A, C]| + |[A, B]| \\ &\leq \omega(X) = c\lambda(G).\end{aligned} \tag{4}$$

Combining inequality (4) with Observation (2), we see that

$$G[B] \text{ is acyclic.} \tag{5}$$

If $G[A]$ contains a cycle, by Observation (2), we have

$$\omega(X \cap Y) = \omega(A) > c\lambda(G). \tag{6}$$

If $G[A]$ is acyclic, since $X = A \cup B$, we see that (6) also holds by property (5), Lemma 5.3, and inequality (4).

Next, we show that

$$\omega(X \cup Y) \geq c\lambda(G). \tag{7}$$

Since $\omega(X \cup Y) = \omega(D)$, inequality (7) is equivalent to $\omega(D) \geq c\lambda(G)$. This is obvious if $G[D]$ contains a cycle. In the following we assume that

$$G[D] \text{ is acyclic.} \tag{8}$$

If $\overline{Y}$ is a good fragment, note that $\overline{Y} = B \cup D$, it follows from (4), (5), (8), and Lemma 5.3 that $\omega(D) > c\lambda(G)$. Hence in the following, we assume that

$$\overline{Y} \text{ is not good.} \tag{9}$$

By Lemma 3.3(i), $\delta(G[\overline{Y}]) \geq 2$. We consider two cases.

Case 1. $\delta(G[\overline{Y}]) \geq 3$.

If $g(G[\overline{Y}]) \geq 6$, then by Corollary 3.2, $G[\overline{Y}]$ contains two disjoint cycles, and thus $|V_q(\overline{Y})| \geq g$ and $|V_p(\overline{Y})| \geq g$, contradicting Assumption (9). Hence, $g(G[\overline{Y}]) = 4$. Then it follows from Assumption (9) that either $|V_q(\overline{Y})| \leq 3$ or $|V_p(\overline{Y})| \leq 3$. Since $\delta(G[\overline{Y}]) \geq 3$, we see that $G[\overline{Y}] \cong K_{3,m}$ for $m = |\overline{Y}| - 3$. Since X is good, we have $|X| \geq 2g$. Then by $|\overline{Y}| \geq |X|$, we have $m \geq |X| - 3 \geq 2g - 3 = 5$.

If $|V_q(\overline{Y})| = 3$, then by $p > q \geq m \geq 5$, we have

$$\begin{aligned}\omega(\overline{Y}) &= mp + 3q - 2 \times 3m \\ &= (p+q-4) \times 2 + (m-2)p + q - 6m + 8 \\ &\geq \zeta(G) + (m-2)(m+1) + m - 6m + 8 \\ &> \zeta(G) > c\lambda(G),\end{aligned}$$

a contradiction. Hence, suppose $|V_p(\overline{Y})| = 3$. Applying Lemma 3.3(ii) to the fragment $\overline{Y}$, we see that every vertex $v \in V_p(\overline{Y})$ has $m = d_{\overline{Y}}(v) \geq d_Y(v)$, and thus $p = d_Y(v) + d_{\overline{Y}}(v) \leq 2m$. Since $G[B]$ and $G[D]$ are acyclic, we have $|E(G[B])| + |E(G[D])| \leq (|B| - 1) + (|D| - 1) = |\overline{Y}| - 2 = (3 + m) - 2 = m + 1$. Note that $q \geq 4$. We have

$$\begin{aligned}\omega(B) + \omega(D) &= 3p + mq - 2(|E(G[B])| + |E(G[D])|) \\ &\geq 3p + mq - 2(m+1)\end{aligned}$$

$$\begin{aligned}
&= 4(p+q-4) + (-p + (m-4)q - 2m + 14) \\
&\geq 4(p+q-4) + (-2m + 4(m-4) - 2m + 14) \\
&= 2\zeta(G) - 2 \geq 2c\lambda(G).
\end{aligned}$$

By inequality (4), $\omega(D) \geq 2c\lambda(G) - \omega(B) \geq c\lambda(G)$.

Case 2. $\delta(G[\overline{Y}]) = 2$.

If $|\overline{Y}| = |Y|$, then $\overline{Y}$ is also an atom, and thus $\delta(G[\overline{Y}]) \geq 3$ by Lemma 3.3(iv), a contradiction. So in this case, $|\overline{Y}| > |Y|$.

Let v be a vertex in $\overline{Y}$ with $d_{\overline{Y}}(v) = 2$. If $G[\overline{Y} - v]$ is acyclic, then similarly to the proof of Lemma 3.3(iii), we see that $G[\overline{Y}]$ is a cycle. Since $|\overline{Y}| > g$, we have $\omega(\overline{Y}) > \zeta(G)$, a contradiction. So $G[\overline{Y} - v]$ contains a cycle, and thus

$$\begin{aligned}
c\lambda(G) &\leq \omega(\overline{Y} - v) = \omega(\overline{Y}) - (d_G(v) - d_{\overline{Y}}(v)) + d_{\overline{Y}}(v) \\
&\leq \omega(\overline{Y}) - \delta(G) + 4 \leq \omega(\overline{Y}) = c\lambda(G),
\end{aligned}$$

which implies that $\overline{Y} - v$ is also a fragment.

Denote $\widetilde{Y} = \overline{Y} - v$. Suppose $\delta(G[\widetilde{Y}]) \geq 3$. Let $\widetilde{B} = B - v$ and $\widetilde{D} = D$ if $v \in B$, and $\widetilde{B} = B$ and $\widetilde{D} = D - v$ if $v \in D$. Note that all the three crucial conditions in deducing Case 1 also hold for $\widetilde{Y} = \widetilde{B} \cup \widetilde{D}$:

(a) $\widetilde{Y}$ is not good;
(b) $G[\widetilde{B}]$ and $G[\widetilde{D}]$ are acyclic;
(c) $\omega(\widetilde{B}) \leq c\lambda(G)$. In fact, this is obviously true if $\widetilde{B} = B$. If $\widetilde{B} = B - v$, then by noting that $d_B(v) \leq d_{\overline{Y}}(v) = 2$, we have

$$\omega(\widetilde{B}) = \omega(B) - (d_G(v) - d_B(v)) + d_B(v) \leq \omega(B) - \delta(G) + 4 \leq \omega(B) \leq c\lambda(G).$$

Then by a similar argument as in Case 1, it can be proved that $\omega(\widetilde{D}) \geq c\lambda(G)$. Similar to the proof of (c) in the above, we have $\omega(\widetilde{D}) \leq \omega(D)$. Hence $\omega(X \cup Y) = \omega(D) \geq c\lambda(G)$.

If $\delta(G[\widetilde{Y}]) = 2$, then the above process continues. Since G is finite, this process terminates at some stage when $\delta(G[\widetilde{Y}]) \geq 3$ for some $\widetilde{Y} \subset \overline{Y}$, and inequality (7) can be obtained at this stage.

Combining inequalities (6) and (7) with the submodular inequality, we have

$$2c\lambda(G) < \omega(X \cap Y) + \omega(X \cup Y) \leq \omega(X) + \omega(Y) = 2c\lambda(G).$$

This contradiction completes the proof of the lemma. □

The following corollary shows that in addition to the conditions in Lemma 5.4, if the girth $g(G) \geq 6$, then any distinct atoms are disjoint.

Corollary 5.5. *Let G be a cyclically separable (p, q)-biregular graph with $p > q \geq 4$. Suppose G is not cyclically optimal and $g(G) \geq 6$. Then for any two distinct atoms X and Y, $X \cap Y = \emptyset$.*

Proof. By Theorem 5.2, X is non-trivial. By Lemma 3.3 (iv), $\delta(G[X]) \geq 3$. Since $g(G) \geq 6$, by Corollary 3.2, $G[X]$ contains two disjoint cycles. Hence X is good. Similarly, Y is good. Then the result follows from Lemma 5.4. □

Now, we are ready to prove the main theorem in this section.

Theorem 5.6. *Let G be a connected edge-transitive graph with order $n \geq 6$ and $\delta(G) \geq 4$. Then G is cyclically optimal.*

Proof. By Lemma 5.1, G is either vertex-transitive or a (p, q)-biregular graph, or we can say that G is either k-regular or (p, q)-biregular with $p > q$. By Corollary 3.2, G is cyclically separable. Suppose G is not cyclically optimal. Then every atom is non-trivial. We are to derive a contradiction.

If any two distinct atoms of G are disjoint, then every atom is an imprimitive block of G, and thus is an independent set by Lemma 4.4, which contradicts the definition of cyclic edge-cut. Hence in the following, we assume that there exist two distinct atoms X and Y such that $X \cap Y \neq \emptyset$. By Corollaries 4.3 and 5.5, this is possible only when $g(G) = 3$ or $g(G) = 4$.

Case 1. $g(G) = 3$. In this case G is a k-regular vertex-transitive graph. By Lemma 3.3(iv), $\delta(G[X]) \geq 3$ and $\delta(G[Y]) \geq 3$. By Lemma 4.2, $|X| = |Y| \leq 2(g(G) - 1) = 4$. It follows that $G[X] \cong G[Y] \cong K_4$. By Lemma 4.2, $|X \cap Y| \leq g(G) - 1 = 2$.

We claim that $k \geq 6$. If $|X \cap Y| = 1$, let u be the unique vertex in $X \cap Y$. Then we see from $G[X] \cong G[Y] \cong K_4$ that $k \geq d_{X \cup Y}(u) = 6$. If $|X \cap Y| = 2$, let $X \cap Y = \{u_1, u_2\}$. Then u_1u_2 is an edge common to two K_4's. By the edge-transitivity of G, every edge is common to two K_4's. It follows that the two ends of any edge have at least 4 common neighbors. Since $G[\overline{X}]$ contains a cycle, we see that $\overline{X \cup Y} \neq \emptyset$. Let z be a vertex in $\overline{X \cup Y}$ which is adjacent to a vertex in $X \cup Y$. If z is adjacent with u_i ($i \in \{1, 2\}$), then $k \geq d_{X \cup Y \cup \{z\}}(u_i) = 6$. Otherwise, let x be a vertex in $X \cup Y$ adjacent with z. Since z is not adjacent with u_1, u_2, we see that x has four neighbors different from u_1, u_2 which are also neighbors of z. Hence $k \geq d_G(x) \geq 7$. The claim is proved.

Now, for any vertex $u \in X$, $d_X(u) = 3 \leq k - d_X(u) = d_{\overline{X}}(u)$, which contradicts Lemma 3.3(iv).

Case 2. G is k-regular with $g(G) = 4$. By Lemma 3.3(iv), $\delta(G[X]) \geq 3$ and $\delta(G[Y]) \geq 3$. By Lemma 4.2, $|X| = |Y| \leq 2(g(G) - 1) = 6$. It follows that $G[X] \cong G[Y] \cong K_{3,3}$.

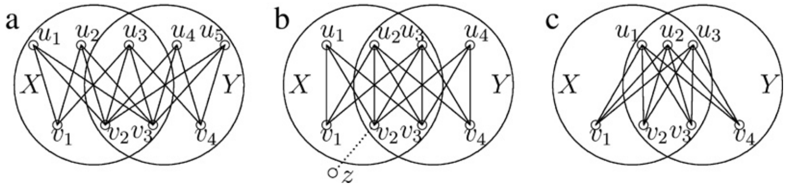

Fig. 1. The only three cases for Case 3.1 when $|V_p(X \cap Y)| = 2$.

By Lemma 4.2, $|X \cap Y| \le g(G) - 1 = 3$. Since $g(G) = 4$, we see that $X \cap Y$ contains a vertex u with $d_{X\cap Y}(u) \le 1$. Hence $k \ge d_{X\cup Y}(u) = d_X(u) + d_Y(u) - d_{X\cap Y}(u) \ge 5$. Then

$$\omega(X) = k|X| - 2|E(G[X])| = 6k - 18 \ge 4(k-2) = \zeta(G) > c\lambda(G),$$

a contradiction.

Case 3. G is (p, q)-biregular with $p > q$ and $g(G) = 4$. If X and Y are good atoms, then by Lemma 5.4, $X \cap Y = \emptyset$, a contradiction. Thus $|V_p(X)| \le 3$ or $|V_q(X)| \le 3$, and $|V_p(Y)| \le 3$ or $|V_q(Y)| \le 3$. Since $\delta(G[X]) \ge 3$ and $\delta(G[Y]) \ge 3$, we have $G[X] \cong G[Y] \cong K_{3,m}$ for some $m \ge 3$.

We claim that the degree-3 parts of the two $K_{3,m}$'s belong to the same part of G, that is, if we denote the bipartition of $G[X]$ by $(U_X^{(3)}, U_X^{(m)})$ and the bipartition of $G[Y]$ by $(U_Y^{(3)}, U_Y^{(m)})$, where $|U_X^{(3)}| = |U_Y^{(3)}| = 3$ and $|U_X^{(m)}| = |U_Y^{(m)}| = m$, then either $U_X^{(3)} \cup U_Y^{(3)} \subseteq V_p(G)$ or $U_X^{(m)} \cup U_Y^{(m)} \subseteq V_p(G)$. Otherwise, without loss of generality, suppose that $U_X^{(3)} \subset V_p(G)$ and $U_Y^{(3)} \subset V_q(G)$. Then $\omega(X) = 3p + mq - 6m$ and $\omega(Y) = mp + 3q - 6m$. By $\omega(X) = \omega(Y)$ and $p > q$, we have $m = 3$. Then $G[X] \cong G[Y] \cong K_{3,3}$, the above claim still holds. As a consequence of this claim, $d_X(v) = d_Y(v)$ for any vertex $v \in X \cap Y$.

If $G[X \cap Y]$ contains an isolated vertex v, then $d_{\overline{X}}(v) \ge d_Y(v) = d_X(v)$, contradicting Lemma 3.3(iv). Hence $V_p(X \cap Y) \ne \emptyset$ and $V_q(X \cap Y) \ne \emptyset$.

Case 3.1. $|V_q(X)| = 3$. In this case, $d_X(v) = 3$ for any vertex v in $V_p(X)$. By Lemma 3.3(iv), $d_{\overline{X}}(v) < d_X(v) = 3$. Hence $p = d_X(v) + d_{\overline{X}}(v) \le 5$. Since $p > q \ge 4$, we have $p = 5$ and $q = 4$.

Let u be a vertex in $V_q(X \cap Y)$. Note that $V_p(X \cup Y) \subseteq N_G(u)$, we have $4 = d_G(u) \ge |V_p(X \cup Y)| = 2m - |V_p(X \cap Y)|$. It follows that $4 \ge |V_p(X \cup Y)| \ge |V_p(X \cap Y)| \ge 2m - 4 \ge 2$.

If $|V_p(X \cap Y)| = 2$, then according to whether $|V_q(X \cap Y)| = 1$ or 2 or 3, $G[X \cup Y]$ is isomorphic to one of the three cases depicted in Fig. 1. In Case (a), we see that u_3v_2 and u_3v_3 are two edges with $N_G(v_2) = N_G(v_3) = V_q(X \cup Y)$ since $p = 5$. For the edge u_3v_1, by the edge-transitivity of G, there must exist an edge u_3v such that $N_G(v_1) = N_G(v)$. Note that v can only be in $\{v_2, v_3, v_4\}$, and all these vertices are adjacent with both u_4 and u_5. Hence $N_G(v_1) = V_q(X \cup Y)$. But then $d_{\overline{Y}}(v_1) = 2 < 3 = d_Y(v_1)$, contradicting Lemma 3.3(iii) (applied to the $c\lambda$-fragment $\overline{Y}$, note that $G[\overline{Y}]$ is not a cycle since otherwise $\omega(\overline{Y}) > \zeta(G) > c\lambda(G)$. In Case (b), u_2 and u_3 are two neighbors of v_2 with $N_G(u_2) = N_G(u_3) = V_p(X \cup Y)$. Since $p = 5$, we see that v_2 has a neighbor $z \in \overline{X \cup Y}$. By the edge-transitivity of G, there is an edge v_2u such that $N_G(z) = N_G(u)$. Note that u can only be in $\{u_1, u_2, u_3, u_4\}$. If $u \in \{u_1, u_2, u_3\}$, then $\{v_1, v_2, v_3\} \subset N_G(z)$. By $d_G(z) = q = 4$, we see that $d_{\overline{X}}(z) = 1$. Similarly, if $u = u_4$, then $d_{\overline{Y}}(z) = 1$. Both contradict Lemma 3.3(i). In Case (c), $d_{\overline{Y}}(v_1) = 2 < 3 = d_Y(v_1)$, contradicting Lemma 3.3(iii).

If $|V_p(X \cap Y)| = 3$, then by $q = 4$, we have $V_p(X \setminus Y) = V_p(Y \setminus X) = \emptyset$. In this case, $V_q(X \setminus Y) \ne \emptyset$. Let u be a vertex in $V_q(X \setminus Y)$. Then $d_{\overline{Y}}(u) = 1$, contradicting Lemma 3.3(i).

Similarly, in the case $|V_p(X \cap Y)| = 4$, we have $d_{\overline{Y}}(u) = 0$ for $u \in V_q(X \setminus Y)$, also a contradiction.

Case 3.2. $|V_p(X)| = 3$. In this case, $p \ge m$. By Lemma 3.3(iv), it can be deduced that $q \le 5$ and $p \le 2m - 1$. Since the case $m = 3$ is reduced to Case 3.1, we may assume $m \ge 4$ in the following.

Let v be a vertex in $V_q(X \cap Y)$. Then by $q = d_G(v) \ge |V_p(X \cup Y)| = 6 - |V_p(X \cap Y)|$, we have

$$3 = |V_p(X)| \ge |V_p(X \cap Y)| \ge 6 - q. \tag{10}$$

Let u be a vertex in $V_p(X \cap Y)$. Then by $p = d_G(u) \ge |V_q(X \cup Y)| = 2m - |V_q(X \cap Y)|$, we have

$$m = |V_q(X)| \ge |V_q(X \cap Y)| \ge 2m - p. \tag{11}$$

If $p \ge m + 2$ and $q \ge 5$, then we have a contradiction that

$$\begin{aligned}\omega(X) &= 3p + mq - 6m \\ &= 2(p + q - 4) + (p + (m-2)q - 6m + 8) \\ &\ge 2(p + q - 4) + (m + 2 + 5(m-2) - 6m + 8) \\ &= 2(p + q - 4) = \zeta(G) > c\lambda(G).\end{aligned}$$

Hence we are left with only two cases:

(i) $q = 5$ and $p = m$ or $m + 1$;
(ii) $q = 4$ and $m \le p \le 2m - 1$.

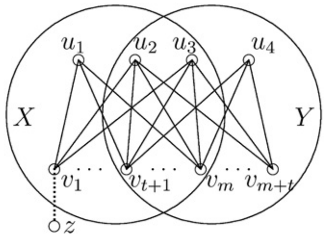

Fig. 2. Labeling vertices for Case (ii).

In case (i), if $p = m$, then $|V_q(X \cap Y)| = m$ by (11), and it can be deduced that $d_{\overline{Y}}(u) = 0$ for any $u \in V_p(X \setminus Y)$. If $p = m + 1$, then $m \geq |V_q(X \cap Y)| \geq m - 1$, and for any $u \in V_p(X \setminus Y)$, either $d_{\overline{Y}}(u) = 1$ or $d_{\overline{Y}}(u) = 2 < m - 1 = d_Y(u)$ (recall that we have assumed that $m \geq 4$). In any case we arrive at a contradiction to Lemma 3.3.

Next, we consider case (ii). In this case, we have $3 \geq |V_p(X \cap Y)| \geq 2$ by (10). If $|V_p(X \cap Y)| = 3$, then $d_{\overline{Y}}(v) = 1$ for any $v \in V_q(X \setminus Y)$, a contradiction. Hence $|V_p(X \cap Y)| = 2$. Suppose $|V_q(X \setminus Y)| = t$. Label the vertices in $X \cup Y$ as in Fig. 2. Then $v_m u_2$ and $v_m u_3$ are two edges with $|N_G(u_2) \cap N_G(u_3)| \geq m + t$. Since $\delta(G[\overline{Y}]) \geq 2$, we see that v_1 has a neighbor $z \in \overline{X \cup Y}$. For the edge $v_1 z$, by the edge-transitivity of G, there is an edge $v_1 u$ with $|N_G(z) \cap N_G(u)| \geq m + t$. This u must be in $\{u_1, u_2, u_3\}$. Then by Lemma 3.3(iv), $d_{\overline{X}}(u) \leq d_X(u) - 1 = m - 1$. Hence $|N_G(z) \cap X| \geq (m + t) - (m - 1) = t + 1$. Since $|V_q(X \setminus Y)| = t$, we see that $N_G(z) \cap V_q(X \cap Y) \neq \emptyset$. However, every vertex in $V_q(X \cap Y)$ has its neighbor set equal to $\{u_1, \ldots, u_4\}$, and thus cannot have z as its neighbor, a contradiction.

The proof is completed. □

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